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On the transient motion of a contained rotating fluid

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This paper considers the transient motion of a viscous fluid in a container rotating with constant angular velocity. The principal objective is to study the manner in which an arbitrary initial state of motion becomes a rigid rotation. In order to concentrate on the effects of viscosity, only the spherical container is studied here in great detail. A general theory will be presented in a subsequent publication.

Several sources of non-uniform behaviour make the analysis difficult and complex. In particular, there are three important time scales, viscous boundary layers, boundary-layer resonances at critical latitudes and intricate side-wall effects. The basic aproach consists of an expansion procedure by means of which the general inviscid solution is corrected for viscous effects and is made uniformly valid in time through the critical spin-up phase. Uniform validity is effected through the elimination of secular terms with unacceptable growth rates arising from the asymptotic perturbation series.

The interior (inviscid) motion leads to a non-self-adjoint partial differential equation eigenvalue problem with many intriguing properties. The general expansion theorem, orthogonality relationships, and viscous decay factors are deduced and used to solve the arbitrary initial-value problem. It is shown that the depth averaged circulation about circular contours, $x^2+y^2 = r^2$, is extracted from the fluid in the spin-up time scale $T = L(\Omega\nu)^{-\frac{1}{2}}$. This is accomplished by a secondary non-oscillatory convective motion produced by suction into the Ekman layer. The excess circulation not eliminated in this way excites inviscid inertial oscillations which are also caused to decay by the boundary layers in the same time scale. Some very small residual effects decay in the ordinary viscous diffusion time, but all the essential processes are concluded in the much shorter interval. All modal oscillations in a sphere are determined and several specific calculations of frequency and decay rate are made and compared with experimental data. Perhaps the most important of these concerns the mode corresponding to rigid internal motion about another axis which can be produced by impulsively changing the rotation axis of the container. Agreement between theory and experiment is very good in all cases compared thus far.

1. Introduction

We consider herein the flow of a fluid in a container rotating with constant angular velocity and the manner in which an arbitrary initial state of motion is resolved into rigid rotation. Of particular interest is the special case of an already rigidly rotating fluid subjected to an impulsive change in the angular velocity of the container.

Spin-up, or the adjustment solely to a change in magnitude of rotation speed, has been discussed extensively by Greenspan & Howard (1963) (hereafter referred to as G–H). Bondi & Lyttleton (1948) and Charney & Eliassen (1949), also considered aspects of the same problem. These studies show that the Ekman boundary layer plays the dominant role in the spin-up phenomenon by producing a slow secondary convective motion through which all important changes occur. Suction into the boundary layer removes low angular momentum fluid from the interior, replacing it with high angular momentum fluid convected radially inward to conserve mass. The conservation of angular momentum provides for the increase in the angular velocity since the interior motion is essentially inviscid. The fluid entering the boundary layers stretches vortex lines and thereby increases the internal vorticity. The description in the case of spin-down is entirely similar. The spin-up time required to accomplish these changes in angular velocity and vorticity is $T = L(\Omega\nu)^{-\frac{1}{2}}$, where L is a characteristic length, Ω is the rotation speed and ν the viscosity.

Stewartson & Roberts (1963) have examined the fluid motion produced by the precessional rotation of a fluid-filled ellipsoidal container. The significant feature of such problems is the fact that the container walls move the fluid about. The action of viscosity alone is more subtle but remarkably effective, none the less. In order to concentrate solely on the effects of viscosity, we shall, for the most part, consider only the spherical container in great detail, and leave the study of more arbitrary container geometries to a subsequent report. The basic viscous processes are undoubtedly independent of container configuration. Stewartson & Roberts do incorporate viscous processes in their analysis by employing a method of successive corrections of an inviscid solution. This procedure has several serious shortcomings for present purposes; it cannot be used, as developed, for the case of a spherical container and, moreover, does not lead directly to a solution that is uniformly valid in a sufficiently long time interval to include spin-up.

Non-uniformity of one kind or another represents the major mathematical obstacle. The motion involves complicated time-dependent boundary layers which, in themselves, have singular features. In addition, there are three time scales based respectively on the period of rotation, spin-up and viscous diffusion into the interior, and each of these is an order of magnitude different from the others. A form of solution is required that is valid at least through the critical spin-up phase, in order for all the important phenomena to be properly included in the analysis; the choice of method is governed by this end.

2. Formulation

In terms of the following dimensionless quantities (unprimed)

$$\mathbf{r}' = L\mathbf{r}, \quad t' = \Omega^{-1}t, \quad \mathbf{q}' = \epsilon L\Omega\mathbf{q}, \quad p'/\rho' = \tfrac{1}{2}\Omega L^2(x^2+y^2)+\epsilon L^2\Omega^2 p, \quad R = \Omega L^2/\nu$$

($\mathbf{r}'$ being a co-ordinate vector, t' the time, $\mathbf{q}'$ the velocity, p' the pressure, ρ' the density, where $\epsilon\Omega L$ characterizes the initial velocity distribution, $\epsilon \ll 1$), the dimensionless equations of motion are, in a uniformly rotating co-ordinate system,

$$\left.\begin{aligned}\partial \mathbf{q}/\partial t + \epsilon(\mathbf{q}\,.\,\nabla)\,\mathbf{q} + 2\hat{\mathbf{k}}\times\mathbf{q} + \nabla p &= R^{-1}\Delta\mathbf{q},\\ \nabla\,.\,\mathbf{q} &= 0,\end{aligned}\right\} \tag{2.1}$$

where $\mathbf{k}$ is the axis of rotation. The contribution of the non-linear terms can be shown to be relatively unimportant to the spin-up process as compared with that of viscosity. Accordingly, we need consider only the linear problem, and the Rossby number ϵ may be effectively taken as zero. Therefore, the fundamental boundary-value problem is

$$\left.\begin{aligned}\partial \mathbf{q}/\partial t + 2\hat{\mathbf{k}}\times\mathbf{q} + \nabla p &= R^{-1}\Delta\mathbf{q},\\ \nabla\,.\,\mathbf{q} &= 0,\end{aligned}\right\} \tag{2.2}$$

with $\mathbf{q} = 0$ on solid boundaries and $\mathbf{q}(x,y,z,0) = \mathbf{q}_*(x,y,z)$. The initial velocity distribution $\mathbf{q}(x,y,z,0)$ must satisfy the mass conservation law and the boundary condition $\hat{\mathbf{n}}\,.\,\mathbf{q}_* = 0$ at the container wall. In other words, $\mathbf{q}_*$ represents a possible state of fluid motion consistent with geometry and does not necessitate an instantaneous impulsive pressure adjustment. Only two of the three velocity components can be arbitrarily prescribed. Thus, the complete boundary-value problem for this configuration is, in cylindrical co-ordinates (r,ω,z),

$$u_t - 2v = -p_r + R^{-1}(u_{rr} + u_{\omega\omega}/r^2 + u_{zz} + u_r/r - u/r^2 - 2v_\omega/r^2), \tag{2.3}$$

$$v_t + 2u = -p_\omega/r + R^{-1}(v_{rr} + v_{\omega\omega}/r^2 + v_{zz} + v_r/r + 2u_\omega/r^2 - v/r^2), \tag{2.4}$$

$$w_t = -p_z + R^{-1}(w_{rr} + w_{\omega\omega}/r^2 + w_{zz} + w_r/r), \tag{2.5}$$

$$u_r + v_\omega/r + w_z + u/r = 0, \tag{2.6}$$

with the initial conditions $u = u_*$, $v = v_*$, $w = w_*$ at $t = 0$, and $u = v = w = 0$ on the bounding surface. Hereafter, unless otherwise noted, the container shall be a sphere of unit dimensionless radius so that the boundary conditions apply at the surface $\rho = 1\,(r = \cos\theta,\ z = \sin\theta)$. It is assumed that the initial conditions exhibit no boundary-layer behaviour.

Although the formulation is now completed, a few brief remarks are deemed appropriate at this time to provide some insight into the general structure of the problem.

If $\xi = \hat{\mathbf{k}}\,.\,\nabla\times\mathbf{q}$, then it follows from (2.2) that

$$\nabla^2 p = 2\xi, \quad \partial\xi/\partial t - R^{-1}\nabla^2\xi = 2\,\partial w/\partial z, \quad \partial w/\partial t - R^{-1}\nabla^2 w = -\,\partial p/\partial z$$

or, as a single equation

$$(\partial/\partial t - R^{-1}\nabla^2)^2\,\nabla^2 p + 4\,\partial^2 p/\partial z^2 = 0. \tag{2.7}$$

(The boundary condition for the pressure alone in the important case of inviscid motion, $R = \infty$, is

$$\frac{\partial^2}{\partial t^2}\left(r\frac{\partial p}{\partial r} + z\frac{\partial p}{\partial z}\right) + 4z\frac{\partial p}{\partial z} + 2\,\frac{\partial^2 p}{\partial\omega\,\partial t} = 0, \tag{2.8}$$

which is the equivalent of $\mathbf{q}\,.\,\hat{\mathbf{n}} = 0$ on $\rho = 1$.)

The hyperbolic nature of the reduced system, $R = \infty$, and some of its unusual features have been discussed extensively in the literature (Phillips 1963 and Squire 1956) and will not be dwelt upon here. It is sufficient for present purposes to appreciate the order and complexity of the basic equations and to recognize that the possible wave motions are both diffusive and dispersive in character.

Separable solutions of the reduced system, (2.7) with $R = \infty$ and (2.8), exist and are of the form

$$p = \phi(r, z)\, e^{ik\omega} e^{i\lambda t},$$

where k is an integer. The eigenvalue problem for the determination of ϕ and λ is then

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\phi\right) - \frac{k^2}{r^2}\phi + \left(1 - \frac{4}{\lambda^2}\right)\frac{\partial^2 \phi}{\partial z^2} = 0$$

with

$$r\frac{\partial \phi}{\partial r} + \frac{2k}{\lambda}\phi + \left(1 - \frac{4}{\lambda^2}\right) z \frac{\partial \phi}{\partial z} = 0$$

on $\rho = 1$. These solutions represent possible inviscid inertial oscillations which can persist inside the container.

Since it is the core of this entire analysis, the mathematical properties of this intriguing non-self-adjoint boundary-value problem are developed in detail in the next section. Aspects of Poincaré's eigenvalue problem, as the foregoing is known, have been studied previously (Cartan 1922; Lyttleton 1953) in connexion with the stability of rotating liquid masses.

In its barest form, the plan of attack calls for the computation of the viscous corrections to each inviscid mode and the construction of a solution that is uniformly valid in time through the spin-up phase, $t = R^{\frac{1}{2}}$. Thus if $y(r, \omega, z, t)$ represents any of the four functions, u, v, w, p, then a solution will be sought of the form

$$y = Y_{n0}(r, \omega, z)\, e^{s_n t} + R^{-\frac{1}{2}} y_{n1}(r, \omega, z, t) + \ldots$$
$$+ \tilde{y}_{n0}(\zeta, \theta, t) + R^{-\frac{1}{2}} \tilde{y}_{n1}(\zeta, \theta, t) + \ldots$$

with

$$s_n = s_{n0} + R^{-\frac{1}{2}} s_{n1} + \ldots.$$

Here $Y_{n0}(r, \omega, z)\, e^{s_{n0} t}$ is the form of the modal solution appropriate to each dependent variable, the tilde symbol denotes a boundary-layer function, $\zeta = R^{\frac{1}{2}}(1-\rho)$ is the stretched boundary-layer co-ordinate and θ is the polar angle. The parameter s_{n1} is of basic importance, for its real part determines the decay rate of the entire inviscid mode, whereas $\operatorname{Im} s_{n1}$ represents the viscous modification of the fundamental eigenfrequency s_{n0} (denoted previously by $i\lambda$).

The complete development of these ideas constitutes §4 and this brief exposition is intended to provide some overall perspective and a brief outline of the work to follow.

3. The inviscid problem

The equations of motion governing the inviscid interior flow are obtained from (2.3) to (2.6) by setting $R = \infty$. The boundary conditions reduce merely to the requirement that the normal component of velocity $\mathbf{q}\,.\,\hat{\mathbf{n}}$ at the container walls be zero.

To examine the possible inertial oscillations, let

$$(p, u, v, w) = (\phi, U, V, W)\, e^{i(\lambda t + k\omega)}, \tag{3.1}$$

where k is an integer and λ is the eigenvalue; the basic equations then reduce to

$$\tfrac{1}{2} i\lambda U - V = -\tfrac{1}{2}\partial\phi/\partial r, \tag{3.2}$$

$$\tfrac{1}{2} i\lambda V + U = -ik\phi/2r, \tag{3.3}$$

$$\tfrac{1}{2} i\lambda W = -\tfrac{1}{2}\partial\phi/\partial z, \tag{3.4}$$

$$(rU)_r + ikV + (rW)_z = 0. \tag{3.5}$$

The boundary condition remains

$$rU + zW = 0 \quad \text{on} \quad \rho = 1. \tag{3.6}$$

The values $|\lambda| = 0, 2$ are somewhat exceptional possibilities and are discussed now prior to the general theory. Let $\lambda = 0$, in which case it follows from (3.4) that $\phi = \phi(r)$, and hence also that

$$U = U(r) = -ik\phi/2r, \quad V = V(r) = \tfrac{1}{2} d\phi/dr.$$

The substitution of these particular forms into (3.5) yields

$$(rW)_z = 0 \quad \text{or} \quad W = W(r).$$

Thus all functions in this case depend on r alone. However, the boundary condition applied at positions $z = \pm(1-r^2)^{\frac{1}{2}}$ requires that

$$rU(r) + (1-r^2)^{\frac{1}{2}}\, W(r) = 0$$

and

$$rU(r) - (1-r^2)^{\frac{1}{2}}\, W(r) = 0,$$

implying that $U = W = 0$. Therefore $\lambda = 0$ is indeed an eigenvalue with eigenfunction

$$U = 0, \quad V = \tfrac{1}{2} d\phi/dr, \quad W = 0, \tag{3.7}$$

where ϕ is an arbitrary function. This mode will be called the spin-up mode.

Consider now the remaining exception, $|\lambda| = 2$. In this case, the momentum equations reduce to

$$\lambda\phi_r = -2k\phi/r, \quad i\lambda U - 2V = -\phi_r, \quad i\lambda W = -\phi_z,$$

and the integration of the first of these yields

$$\phi = F(z)\, r^{-2k/\lambda},$$

F being an arbitrary function. To be physically acceptable, $\lambda = -2k/|k|$ so that $\phi = F(z)\, r^{|k|}$. The second and third equations of the foregoing set then imply that

$$U - ikV/|k| = -\tfrac{1}{2} ik r^{|k|-1} F(z),$$

$$W = -\tfrac{1}{2}(ik/|k|)\, r^{|k|} F'(z).$$

If U and W are considered as functions of F and V, the mass conservation equation can be used to establish the relationship

$$V = \tfrac{1}{4}[|k|\, r^{|k|-1} F(z) + \{r^{|k|+1}/(|k|+1)\} F''(z)].$$

Since all velocities are now expressed in terms of $F(z)$, the substitution of these expressions in the boundary condition leads to the differential equation

$$(1-z^2) F'' - 2(1+|k|)\, zF' - |k|(|k|+1) F = 0, \tag{3.8}$$

which has the general solution

$$F = A(1-z)^{-|k|}+B(1+z)^{-|k|}. \tag{3.9}$$

The function F is necessarily singular; all velocity components must also be singular at the poles, $r = 0, z = \pm 1$. Therefore $|\lambda| = 2$ does not lead to physically acceptable solutions and is not a proper eigenvalue of the system. These positions are, however, limit points of the eigenvalue spectrum but further discussion of their possible significance will not be given here.

If $\lambda \neq 0, \pm 2$, the velocity components may be eliminated from (3.2) to (3.5) to obtain the fundamental eigenvalue problem

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \phi}{\partial r}-\frac{k^2}{r^2}\phi+\left(1-\frac{4}{\lambda^2}\right)\frac{\partial^2\phi}{\partial z^2} = 0, \tag{3.10}$$

with

$$r\frac{\partial \phi}{\partial r}+\frac{2k}{\lambda}\phi+\left(1-\frac{4}{\lambda^2}\right)z\frac{\partial \phi}{\partial z} = 0 \tag{3.11}$$

on $\rho = 1$.

This system has the following properties:

Property 1. *The eigenvalues are real.* Multiply (3.10) by the conjugate of ϕ, $\phi^{\dagger}$, and integrate over the volume of the sphere;

$$\iint r\,dr\,dz\,\phi^{\dagger}\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \phi}{\partial r}-\frac{k^2}{r^2}\phi+\left(1-\frac{4}{\lambda^2}\right)\frac{\partial^2\phi}{\partial z^2}\right] = 0,$$

or upon integrating by parts

$$-\int_0^{\pi}\sin\theta\,\phi^{\dagger}\left(r\frac{\partial \phi}{\partial r}+(1-4\lambda^{-2})\,z\frac{\partial \phi}{\partial z}\right)\bigg|_{\rho=1}d\theta$$
$$+\iint\left(\left|\frac{\partial \phi}{\partial r}\right|^2+\frac{k^2}{r^2}|\phi|^2\right)r\,dr\,dz+\left(1-\frac{4}{\lambda^2}\right)\iint\left|\frac{\partial \phi}{\partial z}\right|^2 r\,dr\,dz = 0. \tag{3.12}$$

The first integral is simplified by using the boundary conditions and the result is

$$+2kI_1/\lambda+I_2+(1-4/\lambda^2)\,I_3 = 0, \tag{3.13}$$

where

$$I_1 = \int_0^{\pi}\sin\theta\,|\phi|^2 d\theta,\quad I_2 = \iint\left(\left|\frac{\partial \phi}{\partial r}\right|^2+\frac{k^2}{r^2}|\phi|^2\right)r\,dr\,dz,$$
$$I_3 = \iint\left|\frac{\partial \phi}{\partial z}\right|^2 r\,dr\,dz,$$

and every integral is positive. (3.13) is a simple quadratic equation for $2/\lambda$ whose solution is

$$2/\lambda = \tfrac{1}{2}[kI_1 \pm (k^2I_1^2+4I_3(I_2+I_3))^{\frac{1}{2}}]I_3^{-1}, \tag{3.14}$$

and this shows that λ is real since the discriminant is always positive. For axially symmetric motions, $k = 0$, it is an immediate consequence of (3.13) that

$$(1-4/\lambda^2) \leqslant 0 \quad\text{or}\quad |\lambda| \leqslant 2.$$

This eigenvalue bound also holds for the non-symmetric eigenfunctions but the general proof is omitted.

Since the eigenvalues are real, the eigenfunctions are in general proportional to real functions, Φ, or $\phi = A\Phi$, but the constant A may be complex. Therefore the remaining properties need be established only on the basis that the eigenfunctions are real.

Furthermore, if (λ, ϕ) is an eigenvalue-eigenfunction pair corresponding to index k, then $(-\lambda, \phi)$ corresponds to index $-k$. This follows directly from the form of the equations.

Property 2. *Orthogonality.* Let (λ, ϕ), (μ, Ψ), be any two different eigenvalue-eigenfunction pairs corresponding to the same index k. Upon multiplying (3.10) by Ψ and the analogous equation for Ψ by ϕ, subtracting the two and integrating over the volume of the sphere (Green's theorem) it is established that

$$k\int_0^\pi r\phi\Psi\,d\theta - \frac{2}{\mu\lambda}(\lambda+\mu)\iint r\,dr\,dz\frac{\partial\phi}{\partial z}\frac{\partial\Psi}{\partial z} = 0; \tag{3.15}$$

the first integral of this expression is a surface integral in terms of the polar angle θ with $r = \sin\theta$, $z = \cos\theta$ and $\rho = 1$. Note that the eigenvalues appear in the orthogonality relationship, a fact related to the non self-adjointness. An extremely useful symmetric form of this relationship, obtained by some simple manipulations, is

$$\iint r\,dr\,dz\left[\frac{\partial\Psi}{\partial r}\frac{\partial\phi}{\partial r} + \frac{k^2\phi\Psi}{r^2} + \frac{\partial\phi}{\partial z}\frac{\partial\Psi}{\partial z}\left(1+\frac{4}{\lambda\mu}\right)\right] = 0. \tag{3.16}$$

The complete modes expressed in (3.1), which correspond to different values of k, are already orthogonal to each other with respect to an integration over the variable ω, because

$$\int_0^{2\pi} \exp\left[i(k_1+k_2)\,\omega\right]d\omega = 0 \quad \text{if} \quad k_1 \neq k_2.$$

Property 3. *A partial expansion theorem.* As a prelude to a complete expansion theorem, we must be able to determine how a given initial disturbance is distributed among the various modes associated with the same index k. These modes are not orthogonal in the usual sense, and the means of calculating the Fourier coefficients is not obvious. If we confine our attention to the single integer k and denote by (λ_n, Φ_n) a characteristic pair with this index, then the general solution is

$$\phi = \sum_n A_n \Phi_n(r,z); \tag{3.17}$$

the corresponding velocity components are

$$U = -\tfrac{1}{2}i\Sigma A_n(1-\tfrac{1}{4}\lambda_n^2)^{-1}\left(\frac{k}{r}\Phi_n + \tfrac{1}{2}\lambda_n\frac{\partial\Phi_n}{\partial r}\right), \tag{3.18}$$

$$V = \tfrac{1}{2}\Sigma A_n(1-\tfrac{1}{4}\lambda_n^2)^{-1}\left(\tfrac{1}{2}\lambda_n\frac{k}{r}\Phi_n + \frac{\partial\Phi_n}{\partial r}\right), \tag{3.19}$$

$$W = i\Sigma\frac{A_n}{\lambda_n}\frac{\partial\Phi_n}{\partial z}. \tag{3.20}$$

Ordinarily in an initial-value problem two of the three velocity components will be prescribed; the task is now to find the coefficients A_n utilizing (3.16). The answer is provided by a knowledge of the physical quantities required to solve an initial-value problem for it is these quantities, when properly combined, that allow use of the orthogonality relationship. From equation (2.7) with $R = \infty$, it follows that both entities

$$\nabla^2 p = 2\xi \quad \text{and} \quad \partial \nabla^2 p/\partial t = 2\partial w/\partial z$$

must be prescribed at $t = 0$. In addition the requirement $ru + zw = 0$ at $\rho = 1$, for all t, is reducible to the extra initial condition

$$rp_r + zp_z = 2rv \quad \text{on} \quad \rho = 1 \quad \text{for} \quad t = 0.$$

Thus the knowledge at time zero of ξ, $\partial w/\partial z$ in the sphere and v on $\rho = 1$ is sufficient to calculate the Fourier coefficients.

Suppose now that the velocity components U, V hence W are given in (3.18)–(3.20). If the vorticity component (parallel to the rotation axis)

$$\xi = [(rV)_r - ikU]/r$$

is computed, it is found that

$$\xi = 2\Sigma \frac{A_n}{\lambda_n^2} \frac{\partial^2 \Phi_n}{\partial z^2}. \tag{3.21}$$

Therefore
$$\iint \Phi_m \xi r \, dr \, dz = 2\Sigma \frac{A_n}{\lambda_n^2} \iint \Phi_m \frac{\partial^2 \Phi_n}{\partial z^2} r \, dr \, dz$$

or by a simple integration by parts

$$\iint \Phi_m \xi r \, dr \, dz = 2\Sigma \frac{A_n}{\lambda_n^2}\left[\int_0^\pi rz\Phi_m \frac{\partial \Phi_n}{\partial z} d\theta - \iint r \, dr \, dz \frac{\partial \Phi_m}{\partial z}\frac{\partial \Phi_n}{\partial z}\right]. \tag{3.22}$$

But
$$rV|_{\rho=1} = \Sigma A_n \left[\frac{2z}{\lambda_n^2}\frac{\partial \Phi_n}{\partial z} - \frac{k}{\lambda_n}\Phi_n\right]_{\rho=1}$$

so that

$$\int_0^\pi r^2 V \Phi_m d\theta = \Sigma A_n \left[\frac{2}{\lambda_n^2}\int_0^\pi rz\Phi_m \frac{\partial \Phi_n}{\partial z} d\theta - \frac{k}{\lambda_n}\int_0^\pi \Phi_n \Phi_m r \, d\theta\right], \tag{3.23}$$

and by subtracting (3.22) from (3.23)

$$-\iint \xi \Phi_m r \, dr \, dz + \int_0^\pi r^2 V \Phi_m d\theta$$
$$= \Sigma A_n \left[\frac{2}{\lambda_n^2}\iint \frac{\partial \Phi_m}{\partial z}\frac{\partial \Phi_n}{\partial z} r \, dr \, dz - \frac{k}{\lambda_n}\int_0^\pi \Phi_n \Phi_m r \, d\theta\right].$$

One further integration by parts yields

$$\int_0^\pi r^2 V \Phi_m d\theta - \iint \xi \Phi_m r \, dr \, dz$$
$$= \Sigma \frac{A_n}{2}\iint r \, dr \, dz \left[\frac{\partial \Phi_n}{\partial r}\frac{\partial \Phi_m}{\partial r} + \frac{k^2}{r^2}\Phi_n \Phi_m + \frac{\partial \Phi_m}{\partial z}\frac{\partial \Phi_n}{\partial z}\right]. \tag{3.24}$$

Finally from the knowledge of W

$$-\frac{2i}{\lambda_m}\iint r \, dr \, dz \, W \frac{\partial \Phi_m}{\partial z} = 2\Sigma \frac{A_n}{\lambda_n \lambda_m}\iint \frac{\partial \Phi_n}{\partial z}\frac{\partial \Phi_m}{\partial z} r \, dr \, dz. \tag{3.25}$$

The calculation is finished by adding the last two expressions and using (3.16); the final result is

$$\tfrac{1}{2}N_m A_m = \int_0^\pi r^2 V\Phi_m d\theta - \iint \xi\Phi_m r\,dr\,dz - \frac{2i}{\lambda_m}\iint W\frac{\partial\Phi_m}{\partial z} r\,dr\,dz, \tag{3.26}$$

with
$$N_m = \iint r\,dr\,dz\left\{\left(\frac{\partial\Phi_m}{\partial r}\right)^2 + \frac{k^2}{r^2}\Phi_m^2 + \left(1+\frac{4}{\lambda_m^2}\right)\left(\frac{\partial\Phi_m}{\partial z}\right)^2\right\}. \tag{3.27}$$

Alternatively, this can be expressed as

$$\tfrac{1}{2}N_m A_m = \iint r\,dr\,dz\left\{\frac{ik}{r}U\Phi_m + V\frac{\partial\Phi_m}{\partial r} - \frac{2i}{\lambda_m}W\frac{\partial\Phi_m}{\partial z}\right\}. \tag{3.28}$$

The Fourier coefficients are now determined; it is interesting to note that the prescription of two arbitrary functions is required to accomplish the task.

Property 4. *The modal mean circulation is zero.* Again let (λ, ϕ) be a characteristic pair with index k; then the actual angular velocity corresponding to these values is

$$v = \tfrac{1}{2}(1-\tfrac{1}{4}\lambda^2)^{-1}\left(\frac{\partial\phi}{\partial r} + \frac{k\lambda}{2r}\phi\right)e^{ik\omega}e^{i\lambda t}. \tag{3.29}$$

The mean circulation about a contour of constant cylindrical radius r defined by

$$\langle\Gamma\rangle = \frac{(1-r^2)^{-\frac{1}{2}}}{4\pi}\int_0^{2\pi} d\omega\int_{-\sqrt{(1-r^2)}}^{\sqrt{(1-r^2)}} rv\,dz$$

is obviously zero unless $k = 0$. In the latter case, $k = 0$ with $\lambda \neq 0$, the basic equation, (3.10), may be integrated directly with respect to z from $z_B = -(1-r^2)^{\frac{1}{2}}$ to $z_T = (1-r^2)^{\frac{1}{2}}$ to establish that

$$\frac{\partial}{\partial r}\int_{z_B}^{z_T} r\phi_r dz = 0 = \frac{\partial}{\partial r}\int_{z_B}^{z_T} rv\,dz. \tag{3.30}$$

The integration of the last equation shows that

$$\int_{z_B}^{z_T} rv\,dz = 0$$

because this quantity is zero at $r = 0$. Therefore the mean circulation is zero in every mode with non-zero frequency.

The initial value problem formulated at the beginning of this section can now be solved completely. The presentation of further properties of the fundamental eigenvalue problem will be continued after the following short but important digression.

3.1. *The complete expansion theorem and the initial-value problem*

Denote by $(\lambda_{nk}, \Phi_{nk})$ the nth eigenvalue-eigenfunction pair corresponding to index k with $\lambda_{nk} \neq 0$. The form of the eigenfunction when $\lambda = 0$ is that of equation (3.7). Therefore the general separable solution of the inviscid boundary-value problem ((2.3)–(2.6) with $R = \infty$ and $ru + zw = 0$ on $\rho = 1$) is

$$p = \phi_0(r) + \sum_{n,k} A_{nk}\exp\left[i(\lambda_{nk}t + k\omega)\right]\Phi_{nk}, \tag{3.31}$$

$$u(r,\omega,z,t) = -\tfrac{1}{2}i\sum_{n,k} A_{nk}(1-\tfrac{1}{4}\lambda_{nk}^2)^{-1}\left(\frac{k}{r}\Phi_{nk}+\frac{\lambda_{nk}}{2}\frac{\partial\Phi_{nk}}{\partial r}\right)\exp i(\lambda_{nk}t+k\omega), \tag{3.32}$$

$$v(r,\omega,z,t) = v_0(r)+\tfrac{1}{2}\sum_{n,k} A_{nk}(1-\tfrac{1}{4}\lambda_{nk}^2)^{-1}\left(\frac{k\lambda_{nk}}{2r}\Phi_{nk}+\frac{\partial\Phi_{nk}}{\partial r}\right)\exp[i(\lambda_{nk}t+k\omega)], \tag{3.33}$$

$$w(r,\omega,z,t) = i\sum_{n,k}\frac{A_{nk}}{\lambda_{nk}}\frac{\partial\Phi_{nk}}{\partial z}\exp[i(\lambda_{nk}t+k\omega)], \tag{3.34}$$

where $v_0(r) = \frac{1}{2}d\phi_0(r)/dr$ and the summation over the k index includes all integers, both positive and negative. It is necessary to recall at this time that if (λ, Φ) is a characteristic pair corresponding to index k then $(-\lambda, \Phi)$ corresponds to index $-k$. In other words

$$\lambda_{n-k} = -\lambda_{nk}, \quad \Phi_{n-k} = \Phi_{nk}.$$

The initial velocity distribution is arbitrarily prescribed consistent with mass conservation and the rigid wall boundary condition. Therefore at $t = 0$, let $(u, v, w) = (u_*, v_*, w_*)$ so that

$$u_*(r,\omega,z) = -\tfrac{1}{2}i\sum_{n,k} A_{nk}(1-\tfrac{1}{4}\lambda_{nk}^2)^{-1}\left(\frac{k}{r}\Phi_{nk}+\frac{\lambda_{nk}}{2}\frac{\partial\Phi_{nk}}{\partial r}\right)e^{ik\omega}, \tag{3.35}$$

$$v_*(r,\omega,z) = v_0(r)+\tfrac{1}{2}\sum_{n,k} A_{nk}(1-\tfrac{1}{4}\lambda_{nk}^2)^{-1}\left(\frac{\lambda_{nk}}{2}\frac{k}{r}\Phi_{nk}+\frac{\partial\Phi_{nk}}{\partial r}\right)e^{ik\omega}, \tag{3.36}$$

$$w_*(r,\omega,z) = i\sum_{n,k}\frac{A_{nk}}{\lambda_{nk}}\frac{\partial\Phi_{nk}}{\partial z}e^{ik\omega}. \tag{3.37}$$

The arbitrary function $v_0(r)$ can be determined at once by using property 4. Thus

$$\int_0^{2\pi} d\omega \int_{-(1-r^2)^{\frac{1}{2}}}^{(1-r^2)^{\frac{1}{2}}} rv_*(r,\omega,z)\,dz = 4\pi(1-r^2)^{\frac{1}{2}}rv_0(r),$$

or

$$v_0(r) = \frac{1}{4\pi(1-r^2)^{\frac{1}{2}}}\int_0^{2\pi} d\omega \int_{-(1-r^2)^{\frac{1}{2}}}^{(1-r^2)^{\frac{1}{2}}} dz v_*(r,\omega,z) = \langle v_*(r,\omega,z)\rangle. \tag{3.38}$$

The spin-up mode, $\lambda = 0$, is the only mode possessing non-zero mean circulation. For a given distribution, the mean or depth averaged circulation 'excites' the spin-up mode, and is in effect removed from the fluid by the spin-up process detailed in G–H. The residual or zero mean circulation distribution stimulates the remaining infinite set of inertial oscillations, each of which will also be shown to decay in the same time scale.

The generalized Fourier coefficients A_{nk} can be obtained by multiplying the preceding equations by $e^{-ik\omega}$ and integrating over the range of ω, with the result

$$\int_0^{2\pi} u_*(r,\omega,z)e^{-ik\omega}d\omega = -\tfrac{1}{2}i\sum_n A_{nk}(1-\tfrac{1}{4}\lambda_{nk}^2)^{-1}\left(\frac{k}{r}\Phi_{nk}+\frac{\lambda_{nk}}{2}\frac{\partial\Phi_{nk}}{\partial r}\right). \tag{3.39}$$

$$\int_0^{2\pi} v_{**}(r,\omega,z)\,e^{-ik\omega}d\omega = \tfrac{1}{2}\sum_n A_{nk}(1-\tfrac{1}{4}\lambda_{nk}^2)^{-1}\left(\frac{\lambda_{nk}}{2}\frac{k}{r}\Phi_{nk}+\frac{\partial\Phi_{nk}}{\partial r}\right),\ v_{**} = v_*-v_0, \tag{3.40}$$

$$\int_0^{2\pi} w_*(r,\omega,z)\,e^{-ik\omega}d\omega = i\sum_n\frac{A_{nk}}{\lambda_{nk}}\frac{\partial\Phi_{nk}}{\partial z}. \tag{3.41}$$

The partial expansion formulae of property 3, equation (3.28), is used to complete the analysis

$$\tfrac{1}{2}N_{nk}A_{nk} = \int_0^{2\pi} e^{-ik\omega}d\omega \iint r\,dr\,dz\left[\frac{ik}{r}u_*\,\Phi_{nk} + v_{**}\frac{\partial\Phi_{nk}}{\partial r} - \frac{2i}{\lambda_{nk}}w_*\frac{\partial\Phi_{nk}}{\partial z}\right] \quad (3.42)$$

with
$$N_{nk} = \iint r\,dr\,dz\left[\left(\frac{\partial\Phi_{nk}}{\partial r}\right)^2 + \frac{k^2}{r^2}\Phi_{nk}^2 + \left(\frac{\partial\Phi_{nk}}{\partial z}\right)^2\left(1+\frac{4}{\lambda_{nk}^2}\right)\right]. \quad (3.43)$$

Property 5. *Explicit formulae for the eigenvalues and eigenfunctions.* Equation (3.10) is actually Laplace's equation (in cylindrical co-ordinates) once the factor $(1-4\lambda_{nk}^{-2})$ is formally absorbed into the z variable and, as such, it is separable in a modified oblate-spheroidal co-ordinate system. Let

$$r = (4/(4-\lambda_{nk}^2)-\eta^2)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}, \quad z = (4/\lambda_{nk}^2-1)^{\frac{1}{2}}\eta\mu; \quad (3.44)$$

the surface of the sphere is then $\mu = \cos\theta$, $\eta = (4/\lambda_{nk}^2-1)^{-\frac{1}{2}}$. Note that the eigenvalue is involved as part of the transformation.

The separable solutions in these co-ordinates are

$$\Phi_{nk} = P_n^k(\eta/c_{nk})\,P_n^k(\mu) \quad (3.45)$$

where $P_n^k(x)$ is an associated Legendre function, and

$$c_{nk} = (1-\tfrac{1}{4}\lambda_{nk}^2)^{-\frac{1}{2}}.$$

The separation is simple but laborious and reference is made to Stewartson & Roberts (1963) for the essential details of a comparable calculation. Insertion of these modal solutions into the boundary condition (3.11) gives rise to the basic eigenvalue equation

$$kP_n^k(x) = (1-x^2)\,d[P_n^k(x)]/dx \quad (3.46)$$

with $x = \frac{1}{2}\lambda_{nk}$. *It is important to note that for each pair of integers* (n, k) *there are in general several eigenvalues.* Thus there are several eigenfunctions of similar form, and a more complete notation is

$$\Phi_{nmk} = P_n^k(\eta/c_{nmk})\,P_n^k(\mu) \quad (3.47)$$

with $c_{nmk} = (1-\frac{1}{4}\lambda_{nmk}^2)^{-\frac{1}{2}}$. Here λ_{nmk} is the mth eigenvalue associated with the Legendre function $P_n^k(x)$. The complete solution would then involve a triple summation over the indices n, m, k. The index m has a finite range, $m = 1, \ldots, M_n$, and M_n is the total number of solutions of (3.46). For the most part, this notation is not used and it is assumed that some one-to-one correspondence between the index triple (n, m, k) and the index pair (n, k) is made.

Property 6. *The eigenvalue bound.* The eigenvalues are related to the roots of the expression

$$E(x) = (1-x^2)\,d[P_n^k(x)]/dx - kP_n^k(x), \quad x = \tfrac{1}{2}\lambda_{nk}$$

which can be also written as

$$E(x) = (1-x^2)\left[\frac{1+x}{1-x}\right]^{\frac{1}{2}}\frac{d(1-x)^k}{dx}\frac{d^kP_n(x)}{dx^k}.$$

The zeros of $E(x)$ are the same as those of $d[(1-x)^k\,d^kP_n(x)/dx^k]/dx$ except possibly for $x = \pm 1$, i.e. $\lambda_{nk} = \pm 2$ but these are not significant anyway. However,

$d^kP_n(x)/dx^k$ is a polynomial of degree $n-k$ all of whose zeros are in the unit interval $|x| \leqslant 1$. Therefore all n roots of $(1-x)^k d^kP_n(x)/dx^k$ are inside the unit interval and the derivative of this function, a polynomial of degree $n-1$, must have $n-1$ zeros in the same span. Thus all the roots of $E(x)$ are in

$$|x| \leqslant 1 \quad (|\lambda_{nk}| \leqslant 2),$$

proving that the eigenvalues of the problem are less than two in absolute magnitude.

Property 7. *The eigenfunctions of property* 6 *are polynomials.* The definition of the associated Legendre functions may be used to express the eigenfunctions of (3.45) as

$$\Phi_{nk} = Ar^k \partial^{2(k+n)} r^{2n} / \partial\eta^{k+n} \partial\mu^{k+n}, \tag{3.48}$$

where A is some constant. In terms of the transformation given in (3.44),

$$\partial^2\psi/\partial\eta\,\partial\mu = \tfrac{1}{2}\lambda_{nk}c_{nk}z(\psi_{rr}+\psi_r/r)+\psi_{rr}(2/\lambda_{nk}c_{nk}r)\,(r^2+(\tfrac{1}{4}\lambda_{nk}^2c_{nk}^2)\,z^2-c_{nk}^2)$$
$$+2\psi_z/\lambda_{nk}c_{nk}+2z\psi_{zz}/\lambda_{nk}c_{nk};$$

and in particular if $\psi = r^{2n}$, then

$$\psi_{\eta\mu} = Azr^{2n-2},$$

with A a different constant. The degree in r is reduced by two. If $\psi = z^m r^{2n}$ where m and n are integers then

$$\psi_{\eta\mu} = A_0 z^{m+1} r^{2n-2} + A_1 z^{m-1} r^{2n} + A_2 z^{m+1} r^{2n-2} + A_3 z^{m-1} r^{2n-2}; \tag{3.49}$$

A_i are constants we shall leave undetermined. Therefore the differential term in (3.48) must be a polynomial in z and r^2, $P(z, r^2)$, and the entire expression is of the form

$$\Phi_{nk} = r^k P(z, r^2).$$

The eigenfunctions corresponding to non-zero eigenvalues are polynomials.

Strictly speaking, it has not been shown that all eigenfunctions are indeed of the functional form obtained by the separation process. There may be eigenfunctions which remain undiscovered by this procedure, for no completeness theorem is available at present. It seems unlikely that any eigenfunctions exist other than those already deduced, but this is, of course, in the absence of proof, a conjecture.

The eigenvalue spectrum has limit points at ± 2 and although it is denumerable, it is also dense in the interval $|\lambda| \leqslant 2$. The full implications of these observations are being studied at present.

4. Viscous effects

The objective of this section is to determine the viscous modifications of the inviscid modes; the solution is required to be uniformly valid through spin-up. The corrections are also to be suitable for the solution of an initial-value problem in which the initial conditions exhibit no boundary-layer structure. The fundamental equations of motion are (2.3) to (2.6).

The modifications of the inviscid modes having non-zero frequency are considered first; the spin-up mode has been analysed at length in (G–H) but is reexamined next from a slightly different point of view.

Let $y(r, \omega, z, t)$ be any one of the dependent variables u, v, w, p, then the appropriate form of a modal solution incorporating the effect of viscosity is

$$y = Y_{n0}(r, \omega, z)\, e^{s_n t} + R^{-\frac{1}{2}} y_{n1}(r, \omega, z, t) + \ldots + \tilde{y}_{n0}(\zeta, \omega, \theta, t) + R^{-\frac{1}{2}} \tilde{y}_{n1}(\zeta, \omega, \theta, t) + \ldots, \tag{4.1}$$

with

$$s_n = s_{n0} + R^{-\frac{1}{2}} s_{n1} + \ldots, \tag{4.2}$$

where $\zeta = R^{\frac{1}{2}}(1-\rho)$, θ is the polar angle and $\sim$ denotes a boundary-layer function. Here $Y_{n0}(r, \omega, z)$ is the inviscid mode, related to $\Phi_{nk}(r, z)\, e^{ik\omega}$ with $s_{n0} = i\lambda_{nk}$ but in the interests of simplicity the subscript k notation is omitted until the final results are achieved. All boundary-layer functions are zero initially.

The ultimate purpose of this investigation is to compute the complex number s_{n1} which will provide the total decay rate of the mode and its frequency alteration. This parameter is determined by the elimination of secular terms arising in the expansions whose growth rates are too rapid to satisfy the requirement of uniform validity through the spin-up phase, $t = O(R^{\frac{1}{2}})$.

No attempt will be made to compute higher-order effects for several reasons. First, the form of the expansion is almost certainly not in powers of $R^{-\frac{1}{2}}$ beyond the first few terms indicated here. Anomalous effects in the Ekman layers introduce extraneous factors of R in subsequent terms, and the series most probably are generalized or composite asymptotic expansions. Secondly, and with good fortune, the higher-order effects are really very small in most cases of current interest and are not of great importance in describing the principal modifications of a basic flow by the viscous Ekman layer.

The substitution of these expansions, represented by (4.1), into the basic equations and the requisite boundary conditions, leads to the following boundary-value problems which must be solved sequentially:

(A)
$$\partial(\tilde{u}_{n0} \sin\theta + \tilde{w}_{n0} \cos\theta)/\partial\zeta = 0, \tag{A. 1}$$

or

$$\tilde{u}_{n0} \sin\theta + \tilde{w}_{n0} \cos\theta = 0;$$

(B)
$$s_{n0} U_{n0} - 2V_{n0} = -\partial P_{n0}/\partial r, \tag{B. 1}$$

$$s_{n0} V_{n0} + 2U_{n0} = -r^{-1} \partial P_{n0}/\partial\omega, \tag{B. 2}$$

$$s_{n0} W_{n0} = -\partial P_{n0}/\partial z, \tag{B. 3}$$

$$\frac{1}{r}\frac{\partial(rU_{n0})}{\partial r} + \frac{1}{r}\frac{\partial V_{n0}}{\partial\omega} + \frac{\partial W_{n0}}{\partial z} = 0, \tag{B. 4}$$

with

$$U_{n0} \sin\theta + W_{n0} \cos\theta = 0 \quad \text{on} \quad \rho = 1;$$

(C)
$$\partial\tilde{u}_{n0}/\partial t - 2\tilde{v}_{n0} = \sin\theta\, \partial\tilde{p}_{n1}/\partial\zeta + \partial^2\tilde{u}_{n0}/\partial\zeta^2, \tag{C. 1}$$

$$\partial\tilde{v}_{n0}/\partial t + 2\tilde{u}_{n0} = \partial^2\tilde{v}_{n0}/\partial\zeta^2, \tag{C. 2}$$

$$\partial\tilde{w}_{n0}/\partial t = \cos\theta\, \partial\tilde{p}_{n1}/\partial\zeta + \partial^2\tilde{w}_{n0}/\partial\zeta^2, \tag{C. 3}$$

with

$$\tilde{u}_{n0} = -U_{n0} e^{s_n t}, \quad \tilde{v}_{n0} = -V_{n0} e^{s_n t}, \quad \tilde{w}_{n0} = -W_{n0} e^{s_n t} \quad \text{on} \quad \rho = 1$$

and zero initial conditions;

(D)
$$\partial u_{n1}/\partial t - 2v_{n1} = -\partial p_{n1}/\partial r - s_{n1} U_{n0} e^{s_n t} \tag{D. 1}$$

$$\partial v_{n1}/\partial t + 2u_{n1} = -r^{-1} \partial p_{n1}/\partial\omega - s_{n1} V_{n0} e^{s_n t}, \tag{D. 2}$$

$$\partial w_{n1}/\partial t = -\partial p_{n1}/\partial z - s_{n1} W_{n0} e^{s_n t}, \tag{D. 3}$$

$$\frac{1}{r}\frac{\partial}{\partial r} r u_{n1} + \frac{1}{r}\frac{\partial v_{n1}}{\partial\omega} + \frac{\partial w_{n1}}{\partial z} = 0, \tag{D. 4}$$

with $$u_{n1}\sin\theta + w_{n1}\cos\theta = -\frac{1}{\sin\theta}\int_0^\infty d\zeta\left(\frac{\partial\tilde{w}_{n0}}{\partial\theta} - \frac{\partial\tilde{v}_{n0}}{\partial\omega}\right) \tag{D. 5}$$

on $\rho = 1$ (outflow from the boundary layer), and zero initial conditions.

Problem (A) leads to the obvious requirement that there be no $O(1)$ normal velocity component at boundaries. This result has been used before.

Problem (B) for the inviscid interior modes was studied in § 3 (where $s_{n0} = i\lambda_n$).

Problem (C) for the boundary-layer functions reduces to that solved in Appendix 1, upon introducing the new dependent variables

$$Q = \tilde{u}_{n0} + i\tilde{v}_{n0}\cos\theta, \quad q = \tilde{u}_{n0} - i\tilde{v}_{n0}\cos\theta.$$

The corresponding boundary conditions at $\zeta = 0$ are simply

$$\left.\begin{aligned} Q &= -(U_{n0} + iV_{n0}\cos\theta)_{\rho=1}e^{s_n t} = a_n(\theta,\omega)\,e^{s_n t},\\ q &= -(U_{n0} - iV_{n0}\cos\theta)_{\rho=1}e^{s_n t} = b_n(\theta,\omega)\,e^{s_n t}.\end{aligned}\right\} \tag{4.3}$$

The essential boundary-layer functions required for the solution of thc next problem, (D), may be computed at once and are

$$\begin{aligned}\int_0^\infty \tilde{v}_{n0}d\zeta &= (2i\cos\theta)^{-1}\int_0^\infty (Q-q)\,d\zeta\\ &= (2i\cos\theta)^{-1}e^{s_n t}(a_n\gamma_n^{-\frac{1}{2}}\operatorname{erf}\gamma_n^{\frac{1}{2}}t^{\frac{1}{2}} - b_n\beta_n^{-\frac{1}{2}}\operatorname{erf}\beta_n^{\frac{1}{2}}t^{\frac{1}{2}}),\end{aligned} \tag{4.4}$$

$$\begin{aligned}\int_0^\infty \tilde{w}_{n0}d\zeta &= -\tfrac{1}{2}\tan\theta\int_0^\infty (Q+q)\,d\zeta\\ &= -\tfrac{1}{2}\tan\theta\, e^{s_n t}(a_n\gamma_n^{-\frac{1}{2}}\operatorname{erf}\gamma_n^{\frac{1}{2}}t^{\frac{1}{2}} + b_n\beta_n^{-\frac{1}{2}}\operatorname{erf}\beta_n^{\frac{1}{2}}t^{\frac{1}{2}}),\end{aligned} \tag{4.5}$$

where $\gamma_n = s_n + 2i\cos\theta$ and $\beta_n = s_n - 2i\cos\theta$.

The boundary-layer functions are finite for all θ, ω and in fact, to a high degree of approximation, have the simple time dependence $\exp(s_n t)$ for almost all values of θ with t moderately large. The exceptions occur in the immediate vicinities of the critical latitudes $-is_{n0} = \lambda_n = \pm 2\cos\theta$. At these positions the modal frequency λ_n equals twice the component of the rotation vector normal to the boundary; i.e. Coriolis force equals acceleration, and a kind of resonance occurs. Continued forced oscillation of the container at a resonant frequency would produce a radically different type of boundary layer at these latitudes. Bondi & Lyttleton (1953) speak of boundary-layer eruptions because the outflow from these regions, computed on the basis of the ordinary steady boundary-layer theory, is infinite. Roberts & Stewartson's (1963) analysis of these zones includes lateral shear terms in the boundary-layer equations, and shows that the local flow field is not truly singular but of a different order in R. They also conclude that the resultant effects on the interior flow are negligible, i.e. $O(R^{-\frac{1}{2}})$. It seems very likely that the non-linear terms are essential to the proper flow description at the critical latitudes.

This difficulty is somewhat mitigated in the transient problem with zero initial conditions because the actual solution is never just a single simple exponential function of time. Continued viscous diffusion to the interior introduces boundary-layer terms such as error functions of time which reflect a new blance at the critical latitude between acceleration and both Coriolis and viscous forces. In other words, the Coriolis force and viscous shear are in equilibrium over most of the boundary, but it is the acceleration which balances both of these forces separately and independently of each other in the immediate vicinities of the critical latitudes. For these reasons, the local structure of the viscous layer is significantly changed. The velocity functions are finite everywhere ($O(1)$ almost everywhere) but possess a non-uniform behaviour with respect to the large parameter R. Some care must be exercised in the mathematical manipulation involving this factor especially when limits and integrations are interchanged.

Thus 'weak spots' in the boundary layer introduce additional non-uniformities into the problem although the total effect on the interior motion remains small. These complications make uncertain the exact form of the fundamental expansions beyond $R^{-\frac{1}{2}}$ terms. A general expansion theory presumably will involve the concepts of inner and outer variables, matching, etc., in several variables (see Lagerstrom & Cole 1955) but the task seems definitely non-trivial.

The use of the Laplace transform (indicated by a bar notation, $\bar{u}$) allows a reduction of problem (D) to

$$(s^2+4)\,\bar{u}_{n1} = -s\left(\frac{\partial \bar{p}_{n1}}{\partial r}+\frac{s_{n1}}{s-s_n}U_{n0}\right)-\frac{2}{r}\frac{\partial \bar{p}_{n1}}{\partial \omega}-\frac{2s_{n1}}{s-s_n}V_{n0},$$

$$(s^2+4)\,\bar{v}_{n1} = -s\left(\frac{1}{r}\frac{\partial \bar{p}_{n1}}{\partial \omega}+\frac{s_{n1}}{s-s_n}V_{n0}\right)+2\frac{\partial \bar{p}_{n1}}{\partial r}+\frac{2s_{n1}}{s-s_n}U_{n0},$$

$$s\bar{w}_{n1} = -\frac{\partial \bar{p}_{n1}}{\partial z}-\frac{s_{n1}}{s-s_n}W_{n0},$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\bar{u}_{n1})+\frac{1}{r}\frac{\partial}{\partial \omega}\bar{v}_{n1}+\frac{\partial}{\partial z}\bar{w}_{n1} = 0,$$

with $$\bar{u}_{n1}\sin\theta+\bar{w}_{n1}\cos\theta = -\frac{1}{\sin\theta}\int_0^\infty d\zeta\left(\frac{\partial}{\partial\theta}\bar{\tilde{w}}_{n0}-\frac{\partial}{\partial\omega}\bar{\tilde{v}}_{n0}\right)$$

on $\rho = 1$. A single equation for the pressure is

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \bar{p}_{n1}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \bar{p}_{n1}}{\partial \omega^2}+\frac{s^2+4}{s^2}\frac{\partial^2 \bar{p}_{n1}}{\partial z^2} = \frac{4s_{n1}(s+s_{n0})}{s^2 s_{n0}^2(s-s_n)}\frac{\partial^2 P_{n0}}{\partial z^2} \tag{4.6}$$

and the corresponding boundary condition on $\rho = 1$ is

$$r\frac{\partial \bar{p}_{n1}}{\partial r}+\frac{2}{s}\frac{\partial \bar{p}_{n1}}{\partial \omega}+\frac{s^2+4}{s^2}z\frac{\partial \bar{p}_{n1}}{\partial z} = \frac{s^2+4}{s\sin\theta}\int_0^\infty d\zeta\left(\frac{\partial \bar{\tilde{w}}_{n0}}{\partial\theta}-\frac{\partial \bar{\tilde{v}}_{n0}}{\partial\omega}\right)$$
$$-\frac{s_{n1}}{s-s_n}\left(\frac{4}{s^2}\cos\theta\, W_{n0}+\frac{2r}{s}V_{n0}\right). \tag{4.7}$$

In the last formula

$$\frac{\partial}{\partial\theta}\int_0^\infty \bar{\tilde{w}}_{n0}d\zeta$$
$$= -\frac{1}{2(s-s_n)}\frac{\partial}{\partial\theta}\{\tan\theta\,[a_n(\theta,\omega)\,(s+2i\cos\theta)^{-\frac{1}{2}}+b_n(\theta,\omega)\,(s-2i\cos\theta)^{-\frac{1}{2}}]\}. \tag{4.8}$$

$$\int_0^\infty \bar{\tilde{v}}_{n0}d\zeta = \frac{1}{2i(s-s_n)\cos\theta}[a_n(\theta,\omega)\,(s+2i\cos\theta)^{-\frac{1}{2}}-b_n(\theta,\omega)\,(s-2i\cos\theta)^{-\frac{1}{2}}]. \tag{4.9}$$

Now each inhomogeneous term has a simple pole at $s = s_n$ (reflecting the fact that $e^{s_n t}$ is the principal time behaviour of all functions over most of the domain), whereas the existence of a homogeneous solution implies that a singularity also occurs at $s = s_{n0} = i\lambda_n$. The general solution for $\bar{p}_{n1}$ must then have two simple poles lying in close proximity to each other at $s = s_{n0}$ and $s = s_n$, a distance $|s_{n1}R^{-\frac{1}{2}}|$ apart. The net effect of such a juxtaposition upon inverting the transforms is to produce a growth rate for p_{n1} which is $O(t)$ when $t \sim R^{\frac{1}{2}}$. This fact can be established most simply by considering the limiting value $R = \infty$, in which case the two simple poles coalesce into a single double pole. The function p_{n1} then would have as its dominant term, $te^{s_n t}$, so that

$$R^{-\frac{1}{2}}p_{n1}e^{-s_n t} = O(R^{-\frac{1}{2}}t) + O(R^{-\frac{1}{2}}t^{\frac{1}{2}}) + \dots,$$

or in general $$R^{-\frac{1}{2}}p_{n1}e^{-s_n t} = O(1) + O(R^{-\frac{1}{4}}) \quad \text{for} \quad t = R^{\frac{1}{2}}.$$

However, the expansions represented by (4.1) are to be uniformly valid throughout the spin-up phase and this requires $R^{-\frac{1}{2}}p_{n1}$ to remain small compared to $P_{n0}e^{s_n t}$ for all $t \leqslant O(R^{\frac{1}{2}})$. Therefore, those terms contributing to the first ordering factor above must be eliminated through the choice of the parameter s_{n1}. (Terms which are $O(R^{-\frac{1}{2}}t^{\frac{1}{2}})$ fulfil the requirement by remaining small at spin-up since they represent the effects of viscous diffusion, and are important only when $t = O(R)$.) The parameter s_{n1} is chosen to eliminate the difficult terms which are exactly those arising from the two neighbouring simple poles. This procedure is obviously entirely similar to the elimination of secular terms in classical perturbation theory.

The fact that the two poles actually lie a short distance $O(R^{-\frac{1}{2}})$ apart does not significantly alter the argument, for a large residue $O(R^{\frac{1}{2}})$ then replaces a time growth that is $O(t)$.

The proper choice of s_{n1} is determined by assuming that $\bar{p}_{n1}$ can have only a simple pole in the complex plane in the immediate vicinity of s_{n0}, whose correct location is $s = s_n$. Therefore, let

$$\bar{p}_{n1} = P_{n1}(r, \omega, z)/(s - s_n) + \text{function regular in the neighbourhood of } s_n.$$

Upon multiplying (4.6) and (4.7) by the factor $(s - s_n)$ and taking the limit $s \to s_{n0} = i\lambda_n$, we find that P_{n1} is a solution of

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial P_{n1}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 P_{n1}}{\partial \omega^2} + \frac{\lambda_n^2 - 4}{\lambda_n^2}\frac{\partial^2 P_{n1}}{\partial z^2} = \frac{8is_{n1}}{\lambda_n^3}\frac{\partial^2 P_{n0}}{\partial z^2} \tag{4.10}$$

and on $\rho = 1$,

$$r\frac{\partial P_{n1}}{\partial r} + \frac{2}{i\lambda_n}\frac{\partial P_{n1}}{\partial \omega} + \frac{\lambda_n^2 - 4}{\lambda_n^2}z\frac{\partial P_{n1}}{\partial z} = \frac{4 - \lambda_n^2}{i\lambda_n \sin\theta}\lim_{s \to s_{n0}}\left\{(s - s_n)\int_0^\infty d\zeta\left[\frac{\partial \bar{\tilde{w}}_{n0}}{\partial \theta} - \frac{\partial \bar{\tilde{v}}_{n0}}{\partial \omega}\right]\right\}$$
$$+ s_{n1}\left[\frac{4}{\lambda_n^2}\cos\theta\, W_{n0} - \frac{2r}{i\lambda_n}V_{n0}\right]. \tag{4.11}$$

The limit process is to be taken in two steps as

$$\lim_{s \to s_{n0}} = \lim_{R \to \infty}\lim_{s \to s_n}$$

and the second limit, $R \to \infty$, may be interchanged with the integral sign occurring in (4.11) only when this is legitimate. In this manner, singular integrals are avoided that arise from the replacement of

$$(i\lambda_n + s_{n1} R^{-\frac{1}{2}} \pm 2i\cos\theta)^{-\frac{1}{2}} \quad \text{by} \quad (i\lambda_n \pm 2i\cos\theta)^{-\frac{1}{2}}.$$

A non-trivial solution P_{n0} of the homogeneous form of the preceding problem exists. The value of s_{n1} for which the inhomogeneous boundary-value problem can have a solution is obtained by multiplying (4.10) by $P^\dagger_{n0}$ and integrating over the volume of the sphere, utilizing the boundary conditions when necessary. In this way, it is established that

$$\int_0^{2\pi} d\omega \int_0^{\pi} d\theta P^\dagger_{n0} \left\{ \frac{4-\lambda_n^2}{i\lambda_n} \lim_{s \to s_{n0}} \int_0^{\infty} d\zeta \left[\frac{\partial}{\partial\theta}(s-s_n)\,\bar{\tilde{w}}_{n0} - \frac{\partial}{\partial\omega}(s-s_n)\,\bar{\tilde{v}}_{n0} \right] \right.$$
$$\left. + s_{n1}\sin\theta\,(4\lambda_n^{-2}\cos\theta\, W_{n0} + 2i\lambda_n^{-1} V_{n0}\sin\theta) \right\}$$
$$= 8i s_{n1}\lambda_n^{-3} \iiint P^\dagger_{n0} \frac{\partial^2 P_{n0}}{\partial z^2}\, r\,dr\,dz\,d\omega.$$

One further integration by parts with respect to θ in the first integrand ($\bar{\tilde{w}} = 0$ at $\theta = 0, \pi$) leads to the result

$$s_{n1} = -iJ^{-1}\lambda_n(4-\lambda_n^2) \int_0^{2\pi} d\omega \int_0^{\pi} d\theta \left[\frac{\partial P^\dagger_{n0}}{\partial\theta} \int_0^{\infty} d\zeta \lim_{s\to s_{n0}} (s-s_n)\,\bar{\tilde{w}}_{n0} \right.$$
$$\left. + P^\dagger_{n0} \int_0^{\infty} d\zeta \lim_{s\to s_{n0}} (s-s_n) \frac{\partial \bar{\tilde{v}}_{n0}}{\partial\omega} \right], \quad (4.12)$$

with
$$J = \int_0^{2\pi} d\omega \int_0^{\pi} d\theta P^\dagger_{n0} \sin\theta \{4\cos\theta\, W_{n0} + 2i\lambda_n \sin\theta\, V_{n0}\}$$
$$- 8i\lambda_n^{-1} \iiint r\,dr\,dz\,d\omega\, P^\dagger_{n0} \frac{\partial^2 P_{n0}}{\partial z^2}. \quad (4.13)$$

Here the limit operation is interchanged with that of integration because in this form the integrand is not only integrable, it is also continuous as $R \to \infty$. Equation (4.12) can be derived by a more conventional approach based on functions having the complete exponential time dependence $\exp(s_n t)$ everywhere including the boundary layer, but this approach leaves much to be desired and is not discussed here.

The last two equations can be reworked into a final form. If the complete notation

$$P_{n0} = \Phi_{nk} e^{ik\omega}, \quad s_{n0} = i\lambda_{nk}, \quad s_{n1} = s_{nk1}, \quad s_n = i\lambda_{nk} + R^{-\frac{1}{2}} s_{nk1},$$

is now introduced then by the simple but laborious procedure of judicious integration by parts and algebraic manipulation, it can be established that

$$s_{nk1} = -\frac{i(4-\lambda_{nk}^2)}{2N_{nk}} \int_0^{\pi} \sin\theta\, d\theta \left\{ \left(\frac{\partial\Phi_{nk}}{\partial\theta} - \frac{k\Phi_{nk}}{\sin\theta} \right)^2 \gamma_{nk}^{-\frac{1}{2}} (\lambda_{nk} + 2\cos\theta)^{-1} \right.$$
$$\left. + \left(\frac{\partial\Phi_{nk}}{\partial\theta} + \frac{k\Phi_{nk}}{\sin\theta} \right)^2 \beta_{nk}^{-\frac{1}{2}} (\lambda_{nk} - 2\cos\theta)^{-1} \right\}, \quad (4.14)$$

where the integral over θ is a surface integral with $r = \cos\theta$, $z = \sin\theta$, and

$$\left.\begin{aligned} N_{nk} &= \iint \left[\left(\frac{\partial\Phi_{nk}}{\partial r}\right)^2 + \frac{k^2}{r^2}\Phi_{nk}^2 + \left(1+\frac{4}{\lambda_{nk}^2}\right)\left(\frac{\partial\Phi_{nk}}{\partial z}\right)^2\right] r\,dr\,dz, \\ \gamma_{nk}^{\frac{1}{2}} &= [i(\lambda_{nk}+2\cos\theta)]^{\frac{1}{2}}, \quad \beta_{nk}^{\frac{1}{2}} = [i(\lambda_{nk}-2\cos\theta)]^{\frac{1}{2}}. \end{aligned}\right\} \tag{4.15}$$

The last two functions always have positive real parts and it follows that

$$\left.\begin{aligned} \gamma_{nk}^{-\frac{1}{2}}(\lambda_{nk}+2\cos\theta)^{-1} &= 2^{-\frac{1}{2}}[|\lambda_{nk}+2\cos\theta|^{-\frac{1}{2}}(\lambda_{nk}+2\cos\theta)^{-1} - i|\lambda_{nk}+2\cos\theta|^{-\frac{3}{2}}], \\ \beta_{nk}^{-\frac{1}{2}}(\lambda_{nk}-2\cos\theta)^{-1} &= 2^{-\frac{1}{2}}[|\lambda_{nk}-2\cos\theta|^{-\frac{1}{2}}(\lambda_{nk}-2\cos\theta)^{-1} - i|\lambda_{nk}-2\cos\theta|^{-\frac{3}{2}}]. \end{aligned}\right\} \tag{4.16}$$

Therefore $\operatorname{Re} s_{nk1} < 0$ for $|\lambda_{nk}| < 2$ and it is not difficult to show that

$$|\operatorname{Im} s_{nk1}| < |\operatorname{Re} s_{nk1}|.$$

All modes are stable and decay in the spin-up time scale, $t = O(R^{\frac{1}{2}})$.

The integrand of the surface integral is finite and continuous for all values of θ. From the exact form of the eigenfunctions, (3.45), evaluated on the surface of the sphere, it is seen that

$$\left(\frac{\partial\Phi_{nk}}{\partial\theta} \pm \frac{k}{r}\Phi_{nk}\right)^2 = \left[(1-\mu^2)^{\frac{1}{2}}\frac{dP_n^k(\mu)}{d\mu} - \frac{kP_n^k(\mu)}{(1-\mu^2)^{\frac{1}{2}}}\right]^2 \tag{4.17}$$

with $\mu = \cos\theta$. This expression is identically zero at $\mu = \frac{1}{2}\lambda_{nk} = \cos\theta$, for it is merely the basic eigenvalue relationship (3.46). Therefore, at the critical latitudes, the integrand behaves like $|\cos\theta \pm \frac{1}{2}\lambda_{nk}|^{\frac{1}{2}}$.

Using the above results, the decay factor of any mode can be calculated with a modest amount of labour. The interior inviscid mode, (4.1), is then uniformly valid through the spin-up phase.

The effective use of equation (4.14) can be illustrated by applying it to a simple but important problem. Consider a fluid-filled spherical container rotating rigidly about a given axis. At time zero, the direction of the axis of rotation of the container is impulsively changed a small amount. Rigid rotation about an axis other than the rotation axis is, however, a possible inviscid mode in the co-ordinate system moving with the spherical container. In fact, this interior motion corresponds to the particular values

$$P_{n0} = irze^{i\omega}, \quad U_{n0} = ze^{i\omega}, \quad V_{n0} = ize^{i\omega}, \quad W_{n0} = -re^{i\omega}$$

or $k = 1$, $\lambda_n = 1$, $\Phi_n = rz$. The result is

$$s_{n1} = -2{\cdot}62 + 0{\cdot}259i. \tag{4.18}$$

This is identical with the limiting result of Stewartson & Roberts (1963) for the viscous correction to an interior mode of a precessing ellipsoid. Their calculation allowed the replacement of the ellipsoid by a sphere for certain purposes and involved an iteration procedure applied to a single mode requiring the complete solution of the problem at each stage. Essentially, the expression $\exp(-R^{\frac{1}{2}}t)$ appears therein as a power series that must be ultimately reconstituted to obtain a uniformly valid solution. Formula (4.14) offers the correction to any mode in relatively simple form and the analysis which led to it can be extended with a modest amount of difficulty to a consideration of arbitrary configurations. It must be recognized, of course, that these analyses have somewhat different objectives, and no attempt is made here to solve the precessional problem by present methods.

As another example, consider the mode

$$\Phi = z^2 - \tfrac{1}{2}z^4 - 2r^2z^2 + \tfrac{2}{3}r^2 - \tfrac{1}{3}r^4$$

corresponding to $n = 4$, $k = 0$, $m = 1$ in the notation of (3.47). The associated eigenvalue is

$$\lambda = \pm 2(\tfrac{3}{7})^{\frac{1}{2}}$$

and the computation produces the decay factor

$$s_{401} = -3{\cdot}77 + 0{\cdot}433i. \tag{4.19}$$

Experiments performed thus far by W. V. R. Malkus at U.C.L.A., K. E. Aldridge and A. Toomre at M.I.T., and W. G. Wing at the Sperry Gyroscope Co. are as yet unreported in the literature. Private communications indicate that the agreement between theory and experiment relating to frequency and decay rate is very close.

The spin-up mode requires special consideration for its decay factor is known (G–H) to be a function of r of the form $\exp[-\sigma(r)\,R^{-\frac{1}{2}}t]$. This is consistent with the preceding results once it is recognized that

$$v(r) = \alpha(r) \quad \text{for} \quad r_0 < r < r_0 + \Delta r_0,$$
$$v(r) = 0 \qquad \text{elsewhere},$$

is a legitimate eigenfunction corresponding to $\lambda = 0$. As such, the boundary corrections to this mode would lead to a local decay factor $\sigma(r_0)$. The general mode $V(r)$ can be interpreted as a sum (integral) of such step function modes, each with a decay factor depending on its radial position. Thus the above quoted result from (G–H) would be consistent with the procedure invoked in this section.

Of course, the fact that $\lambda = 0$ implies that the significant time scale for this mode is really $O(R^{\frac{1}{2}})$. The spin-up time should then be used as a characteristic value and not the period of rotation. If, now, the more appropriate time scale

$$\tau = R^{-\frac{1}{2}}t \tag{4.20}$$

is introduced into the basic equations (recall that the spin-up mode is axially symmetric), the rescaled problem is

$$\left.\begin{aligned} R^{-\frac{1}{2}}u_\tau - 2v &= -p_r + R^{-1}(u_{rr} + u_{zz} + u_r/r - u/r^2),\\ R^{-\frac{1}{2}}v_\tau + 2u &= R^{-1}(v_{rr} + v_{zz} + v_r/r - v/r^2),\\ R^{-\frac{1}{2}}w_\tau &= -p_z + R^{-1}(w_{rr} + w_{zz} + w_r/r),\\ u_r + u/r + w_z &= 0; \end{aligned}\right\} \tag{4.21}$$

$$u = v = w = 0 \quad \text{on} \quad \rho = 1, \quad u = 0, \quad v = v(r), \quad w = 0, \quad \text{at} \quad t = 0.$$

The analysis now proceeds in an identical manner to that in (G–H) and reference is made to that paper for complete details. Let

$$\left.\begin{aligned} u(r,z,\tau) &= \qquad\qquad R^{-\frac{1}{2}}u_1(r,z,\tau)+\ldots+\tilde{u}_0(\zeta,\theta,\tau)+R^{-\frac{1}{2}}\tilde{u}_1(\zeta,\theta,\tau)+\ldots,\\ v(r,z,\tau) &= v_0(r,\tau)+R^{-\frac{1}{2}}v_1(r,z,\tau)+\ldots+\tilde{v}_0(\zeta,\theta,\tau)+R^{-\frac{1}{2}}\tilde{v}_1(\zeta,\theta,\tau)+\ldots,\\ w(r,z,\tau) &= \qquad\qquad R^{-\frac{1}{2}}w_1(r,z,\tau)+\ldots+\tilde{w}_0(\zeta,\theta,\tau)+R^{-\frac{1}{2}}\tilde{w}_1(\zeta,\theta,\tau)+\ldots,\\ p(r,z,\tau) &= p_0(r,\tau)+R^{-\frac{1}{2}}p_1(r,z,\tau)+\ldots+\tilde{p}_0(\zeta,\theta,\tau)+R^{-\frac{1}{2}}\tilde{p}_1(\zeta,\theta,\tau)+\ldots. \end{aligned}\right\} \tag{4.22}$$

The matching of interior solutions to boundary-layer corrections leads directly to an equation for v_0 alone,

$$\partial v_0(r,\tau)/\partial\tau = -v_0(r,\tau)/(1-r^2)^{\frac{3}{4}},$$

and the final result is

$$v_0 = v(r)\exp\{-(1-r^2)^{-\frac{3}{4}}\tau\} \tag{4.23}$$

with

$$\partial p_0/\partial r = 2v_0(r,\tau).$$

The arbitrary function $v(r)$ appearing in this equation is determined from the initial conditions. An important observation, worth repeating, is that the decay rate for the spin-up mode is a function of radial position. It has already been noted that the spin-up mode is responsible for the extraction of the mean circulation of the initial velocity distribution. The exact form of the small velocity components u_1, w_1 as well as the boundary-layer corrections may be found in the cited reference.

5. The initial-value problem

The results of the preceding sections are gathered together here so that the complete interior solution of the initial-value problem, uniformly valid in time through spin-up, may be displayed in full.

Formulae (3.31) to (3.34) need only be corrected to include the decay factory s_{nk1} computed in the last section. The complete uniformly valid interior solution is

$$u(r,\omega,z,t) = -\tfrac{1}{2}i\sum_{n,k}A_{nk}(1-\tfrac{1}{4}\lambda_{nk}^2)^{-1}\left(\frac{k}{r}\Phi_{nk}+\frac{\lambda_{nk}}{2}\frac{\partial\Phi_{nk}}{\partial r}\right)E_{nk}, \tag{5.1}$$

$$\begin{aligned} v(r,\omega,z,t) &= v(r)\exp(-R^{-\frac{1}{2}}(1-r^2)^{-\frac{3}{4}}t)\\ &\quad+\tfrac{1}{2}\sum_{n,k}A_{nk}(1-\tfrac{1}{4}\lambda_{nk}^2)^{-1}\left(\frac{\lambda_{nk}}{2}\frac{k}{r}\Phi_{nk}+\frac{\partial\Phi_{nk}}{\partial r}\right)E_{nk}, \end{aligned} \tag{5.2}$$

$$w(r,\omega,z,t) = i\sum_{n,k}\frac{A_{nk}}{\lambda_{nk}}\frac{\partial\Phi_{nk}}{\partial z}E_{nk}, \tag{5.3}$$

$$p(r,\omega,z,t) = \phi(r)\exp\{-R^{-\frac{1}{2}}(1-r^2)^{-\frac{3}{4}}t\}+\sum_{n,k}A_{nk}\Phi_{nk}E_{nk}, \tag{5.4}$$

with

$$v(r) = \tfrac{1}{2}d\phi/dr,\quad E_{nk} = \exp[i(\lambda_{nk}t+k\omega)+s_{nk1}R^{-\frac{1}{2}}t].$$

At time $t=0$, the initial conditions, $u_*(r,\omega,z)$, $v_*(r,\omega,z)$, $w_*(r,\omega,z)$ are used to determine the constants A_{nk} and the arbitrary function $v(r)$ as detailed in (3.39)–(3.42). Thus

$$v(r) = \frac{(1-r^2)^{-\frac{1}{2}}}{4\pi}\int_0^{2\pi}d\omega\int_{-(1-r^2)^{\frac{1}{2}}}^{(1-r^2)^{\frac{1}{2}}}dz\,v_*(r,\omega,z), \tag{5.5}$$

$$\tfrac{1}{2}N_{nk}A_{nk} = \int_0^{2\pi} d\omega e^{-ik\omega} \iint r\,dr\,dz \left(\frac{ik\Phi_{nk}u_*}{r} + v_{**}\frac{\partial\Phi_{nk}}{\partial r} - \frac{2i}{\lambda_{nk}} w_* \frac{\partial\Phi_{nk}}{\partial z}\right), \tag{5.6}$$

$$v_{**} = v_* - v(r),$$

$$N_{nk} = \iint r\,dr\,dz \left[\left(\frac{\partial\Phi_{nk}}{\partial r}\right)^2 + \frac{k^2}{r^2}\Phi_{nk}^2 + \left(\frac{\partial\Phi_{nk}}{\partial z}\right)^2\left(1+\frac{4}{\lambda_{nk}^2}\right)\right]. \tag{5.7}$$

Finally, the decay factor is that of equation (4.14)

$$s_{nk1} = -\frac{i(4-\lambda_{nk}^2)}{2N_{nk}}\int_0^{\pi} \sin\theta\,d\theta \left[\left(\frac{\partial\Phi_{nk}}{\partial\theta} - \frac{k\Phi_{nk}}{\sin\theta}\right)^2 \gamma_{nk}^{-\frac{1}{2}}(\lambda_{nk}+2\cos\theta)^{-1} + \left(\frac{\partial\Phi_{nk}}{\partial\theta} + \frac{k\Phi_{nk}}{\sin\theta}\right)^2 \beta_{nk}^{-\frac{1}{2}}(\lambda_{nk}-2\cos\theta)^{-1}\right] \tag{5.8}$$

with
$$\gamma_{nk} = i(\lambda_{nk}+2\cos\theta), \quad \beta_{nk} = i(\lambda_{nk}-2\cos\theta). \tag{5.9}$$

We shall not record the associated boundary-layer functions.

6. Forced oscillations

The response of the fluid to the forced oscillation of the spherical container is another important problem that can now be solved. Suppose for definiteness, the sphere is oscillated at frequency α, then it is of interest to know the induced response (the amplitude) of each natural mode that is excited. The inviscid modes are stimulated by the small mass flux into the viscous boundaries, i.e. the convergence or divergence of the Ekman layers. The boundary condition for the interior flow at the walls is then

$$ru + zw = F(\theta,\omega)\,e^{i\alpha t} \quad \text{at} \quad \rho = 1, \tag{6.1}$$

where $F(\theta,\omega)$ is a known function. Since $F(\theta,\omega) = \sum_k F_k(\theta)\,e^{ik\omega}$, it will be sufficient to consider the response to a single Fourier component, say $F_k(\theta)\,e^{ik\omega}e^{i\alpha t}$. The equivalent problem for the pressure

$$P = \phi(r,z)\,e^{ik\omega}e^{i\alpha t}$$

is
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) - \frac{k^2}{r^2}\phi + \left(1-\frac{4}{\alpha^2}\right)\frac{\partial^2\phi}{\partial z^2} = 0 \tag{6.2}$$

with
$$r\frac{\partial\phi}{\partial r} + \frac{2k}{\alpha}\phi + \left(1-\frac{4}{\alpha^2}\right)z\frac{\partial\phi}{\partial z} = G_k(\theta) \quad \text{on} \quad \rho = 1. \tag{6.3}$$

Here $G_k(\theta)$ is a known function. This general boundary-value problem may be solved by a Fourier expansion involving the modes associated with index k,

$$\phi = \sum_{n=1}^{\infty}\sum_{m=1}^{M_{nk}} A_{nmk}\Phi_{nmk} = \sum_{n,m} A_{nmk}\Phi_{nmk}, \tag{6.4}$$

where the complete notation introduced in (3.47) *et seq.* is utilized. For definite values of n and k, the summation index m varies between one and integer M_{nk}, representing the total number of acceptable eigenvalues arising from (3.46). Recall that $(\lambda_{nmk}, \Phi_{nmk})$ is the mth characteristic pair corresponding to the associated Legendre function $P_n^k(x)$. It is especially important to note, once again, that all the functions Φ_{nmk} (m varying, n, k fixed) reduce to the same zonal harmonic on the surface of the sphere, i.e.

$$\Phi_{nmk}\big|_{\rho=1} = P_n^k(\tfrac{1}{2}\lambda_{nmk})\,P_n^k(\cos\theta). \tag{6.5}$$

The substitution of (6.4) into (6.2) and (6.3), together with use of (3.10) and (3.11), implies that

$$4\sum_{n,m} A_{nmk}\left(\frac{1}{\lambda_{nmk}^2}-\frac{1}{\alpha^2}\right)\frac{\partial^2}{\partial z^2}\Phi_{nmk} = 0, \tag{6.6}$$

$$2\sum_{n,m} A_{nmk}\left[k\left(\frac{1}{\alpha}-\frac{1}{\lambda_{nmk}}\right)\Phi_{nmk}+2\left(\frac{1}{\lambda_{nmk}^2}-\frac{1}{\alpha^2}\right)z\frac{\partial\Phi_{nmk}}{\partial z}\right] = G_k(\theta). \tag{6.7}$$

These may now be multiplied by $r\Phi_{\nu\beta k}$, integrated over the volume and surface of the sphere respectively, and added together to obtain

$$2\sum_{n,m} A_{nmk}\left[2\left(\frac{1}{\lambda_{nmk}^2}-\frac{1}{\alpha^2}\right)\iint\frac{\partial\Phi_{nmk}}{\partial z}\frac{\partial\Phi_{\nu\beta k}}{\partial z}r\,dr\,dz + k\left(\frac{1}{\alpha}-\frac{1}{\lambda_{nmk}}\right)\int_0^\pi\Phi_{nmk}\Phi_{\nu\beta k}r\,d\theta\right] = \int_0^\pi G_k r\,\Phi_{\nu\beta k}\,d\theta. \tag{6.8}$$

Finally, with use of the orthogonality condition (3.15), this in turn is expressible as

$$A_{\nu\beta k}C_{\nu\beta k}-2k\sum_{n,m}{}' A_{nmk}\frac{(\alpha-\lambda_{nmk})(\alpha-\lambda_{\nu\beta k})}{\alpha^2(\lambda_{nmk}+\lambda_{\nu\beta k})}\int_0^\pi\Phi_{nmk}\Phi_{\nu\beta k}r\,d\theta = \int_0^\pi G_k r\Phi_{\nu\beta k}\,d\theta, \tag{6.9}$$

where $$C_{\nu\beta k} = 4\left(\frac{1}{\lambda_{\nu\beta k}^2}-\frac{1}{\alpha^2}\right)\iint\left(\frac{\partial\Phi_{\nu\beta k}}{\partial z}\right)^2 r\,dr\,dz+2k\left(\frac{1}{\alpha}-\frac{1}{\lambda_{\nu\beta k}}\right)\int_0^\pi\Phi_{\nu\beta k}^2 r\,d\theta \tag{6.10}$$

and the primed summation symbol indicates that $(n,m) \neq (\nu,\beta)$. However it is apparent from the particular form of the eigenfunctions (6.5) that

$$\int_0^\pi\Phi_{nmk}\Phi_{\nu\beta k}r\,d\theta = P_n^k(\tfrac{1}{2}\lambda_{nmk})P_\nu^k(\tfrac{1}{2}\lambda_{\nu\beta k})\frac{2}{2n+1}\frac{(n+k)!}{(n-k)!}\delta_{n\nu}.$$

Therefore

$$A_{\nu\beta k}C_{\nu\beta k}-4k\sum_{\substack{m=1\\ m\neq\beta}}^{M_{\nu k}} A_{\nu mk}\frac{(\alpha-\lambda_{\nu mk})(\alpha-\lambda_{\nu\beta k})}{\alpha^2(\lambda_{\nu mk}+\lambda_{\nu\beta k})}\frac{P_\nu^k(\frac{1}{2}\lambda_{\nu mk})P_\nu^k(\frac{1}{2}\lambda_{\nu\beta k})}{(2\nu+1)}\frac{(\nu+k)!}{(\nu-k)!} = \int_0^\pi G_k r\,\Phi_{\nu\beta k}\,d\theta \quad (\beta = 1,2,\ldots,M_{\nu k}). \tag{6.11}$$

There are then $M_{\nu k}$ equations in $M_{\nu k}$ unknowns, $A_{\nu\beta k}$, $\beta = 1,\ldots,M_{\nu k}$, and the system is exactly soluble. The coefficients give the induced modal amplitudes; it is now a straightforward task to solve the most general forced oscillation problem.

In the case of axial symmetry, $k = 0$, the solution is particularly simple because the equations of (6.11) uncouple.

Inspection of this finite set of linear equations indicates that resonance occurs whenever $\alpha = \lambda_{nmk}$, an eigenvalue. Since the eigenvalue spectrum is dense for

$|\lambda| < 2$ resonance always occurs in the linear theory whenever α is in the same range. However, the actual resonant amplitude differs from mode to mode according to their ability to absorb energy; the higher modes are more difficult to excite. Be that as it may, the linear theory must be corrected for the effects of viscosity to properly account for the response at resonance. This may be done simply by using the more precise eigenvalues, those corrected for viscosity to order $R^{-\frac{1}{2}}$, in all of the preceding formulae.

The expansion process developed in this paper is probably the first step towards a complete singular perturbation theory utilizing the concepts of inner and outer variables (Lagerstrom & Cole 1955); however, the existence of multiple time scales, boundary layers, critical latitudes, and side wall effects certainly make the task extremely difficult, to say the least. The lack of a complete expansion theory may provide the incentive for a more systematic investigation even though the results which have been attained seem very good on the whole.

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Appendix 1

General boundary-layer solution

Consider the general boundary-layer problem

$$\left(\frac{\partial^2}{\partial\zeta^2}-\frac{\partial}{\partial t}\right) Q - 2i\cos\theta Q = 0, \tag{I 1}$$

$$\left(\frac{\partial^2}{\partial\zeta^2}-\frac{\partial}{\partial t}\right) q + 2i\cos\theta q = 0, \tag{I 2}$$

and $Q = Q_0 e^{s_0 t}$, $q = q_0 e^{s_0 t}$ for $\zeta = 0$, $Q = q = 0$ when $t = 0$. Of particular importance is the function

$$\bar{\phi}|_{\zeta=0} = \frac{1}{2i}\int_0^\infty (q-Q)\,d\zeta.$$

The solution is easily obtained by taking Laplace transforms

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt,$$

and using the inversion formulas of Foster & Campbell (1948, especially no. 805.3). The results are

$$Q = \tfrac{1}{2}Q_0 e^{s_0 t}[\exp\{-\gamma^{\frac{1}{2}}\zeta\,\mathrm{erfc}\,(\zeta/2t^{\frac{1}{2}}-\gamma^{\frac{1}{2}}t^{\frac{1}{2}})\} + \exp\{\gamma^{\frac{1}{2}}\zeta\,\mathrm{erfc}\,(\zeta/2t^{\frac{1}{2}}+\gamma^{\frac{1}{2}}t^{\frac{1}{2}})\}], \tag{I 3}$$

$$q = \tfrac{1}{2}q_0 e^{s_0 t}[\exp\{-\beta^{\frac{1}{2}}\zeta\,\mathrm{erfc}\,(\zeta/2t^{\frac{1}{2}}+\beta^{\frac{1}{2}}t^{\frac{1}{2}})\} + \exp\{\beta^{\frac{1}{2}}\zeta\,\mathrm{erfc}\,(\zeta/2t^{\frac{1}{2}}+\beta^{\frac{1}{2}}t^{\frac{1}{2}})\}], \tag{I 4}$$

where $\quad \gamma = s_0 + 2i\cos\theta, \quad \beta = s_0 - 2i\cos\theta$

and $\gamma^{\frac{1}{2}}$, $\beta^{\frac{1}{2}}$ have positive real parts.

Of frequent use in the analysis is the result

$$\tilde{\phi}|_{\zeta=0} = -\tfrac{1}{2}ie^{s_0 t}[Q_0\gamma^{-\frac{1}{2}}\operatorname{erf}\gamma^{\frac{1}{2}}t^{\frac{1}{2}} - q_0\beta^{-\frac{1}{2}}\operatorname{erf}\beta^{\frac{1}{2}}t^{\frac{1}{2}}], \tag{I 5}$$

and it is also worth while noting that the Laplace transform of this function is

$$\bar{\tilde{\phi}}|_{\zeta=0} = -\tfrac{1}{2}i[Q_0/(s-s_0)\,(s+2i\cos\theta)^{\frac{1}{2}} - q_0/(s-s_0)\,(s-2i\cos\theta)^{\frac{1}{2}}]. \tag{I 6}$$

Note added in proof

Since this paper was submitted, a general theory of contained rotating fluid motions has been completed. The theoretical analysis of transient flows inside arbitrarily shaped rotating containers is found to be greatly simplified and clarified by using the velocity vector and not the pressure as the quantity of primary importance. The derivations of properties 1, 2, and 3 may then be developed in a more direct manner than indicated here. For example, the orthogonality relationship is really a statement involving the dot product of complex modal vector velocities integrated over the container volume. Other properties have also been extended and it has been shown that the generalization of property 4 concerns mean circulation about geostrophic contours.

However, for specific applications, the actual solution procedure requires solving for the pressure function first, since only then can the velocity components be determined. Thus in any particular problem it would be necessary to convert the general theory, written in terms of the velocity, into the equivalent structure for the pressure alone to achieve the most efficient and simplest means of computation. The present work is then typical in every way of the analysis for any particular container configuration.

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